October 22: Week 7 - Number Theory

Warm up

(a) What kinds of remainders do squares have mod 3?
(b) What kinds of remainders do squares have mod 5?
(c) What kinds of remainders do cubes have mod 9?
(d) What is the largest power of 3 that divides $27! = 27 \cdot 26 \cdot 25 \cdots 2 \cdot 1$?
(e) Simplify $2^{100}3^{20}$ mod 10.
(f) Simplify 67^{24} mod 7. (Hint: Use Fermat's Little Theorem to make this go faster)
(g) Simplify 65^{4321} mod 77. (Hint: Use the Chinese Remainder theorem)

Problem 1

Show that 2002^{2002} cannot be written as a sum of three cubes.

Problem 2

Show that 2^n does not divide $n!$ for any $n \geq 1$.

Problem 3 (Putnam 2007)

Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$.

Problem 4

Show that the system of equations

$$5x^2 + y^2 = z^2$$
$$x^2 + 5y^2 = t^2$$

has no non zero integer solutions.

Problem 5 (Putnam 1984)

For an integer n define $f(n) = 1! + 2! + \cdots + n!$. Find polynomials $P(n)$ and $Q(n)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n) \quad \text{for all } n \geq 1$$

Problem 6

Consider the sequence (a_n) defined recursively by $a_1 = 2$, $a_2 = 5$, and

$$a_{n+1} = (2 - n^2)a_n + (2 + n^2)a_{n-1}$$

for $n \geq 2$. Do there exist indices p, q, r such that $a_p a_q = a_r$?

Problem 7 (Putnam 2017)

Suppose that a positive integer N can be expressed as the sum of k consecutive positive integers

$$N = a + (a+1) + (a+2) + \cdots + (a+k-1)$$

for $k = 2017$ but for no other values of $k > 1$, Considering all positive integers N with this property, what is the smallest positive integer a that occurs in any of these expressions?

Non-number theory bonus problem

In the Senate of Kazakhstan, each member has at most three enemies. A member cannot be his own enemy, and enmity is mutual. Prove that the Senate can be divided into two factions such that each Senator has at most one enemy within his faction.

Many of these problems are from the Fall 2021 Putnam Class at Notre Dame

Hints

Problem 1 What is 2002^{2002} mod 9? What are the possible remainders of cubes mod 9?

Problem 2 How many multiples of 2^k are less than or equal to n?

Problem 3 Try mod $f(n)$.

Problem 4 Add them up and try some modular arithmetic. This is an extension of the warm up problem.

Problem 5 There are many ways to do this. Try finding $P(n)$ mod $(1! + 2! + \cdots n!)$.

Problem 6 Try finding a_n mod 3.

Problem 7 Find a formula for $2N$ as a product of 2-terms and figure out what properties you need for the factorization to be unique.

The Bonus Problem is from https://brilliant.org/wiki/invariant-principle-definition/